On the existence of D–solutions of the steady–state Navier–Stokes equations in plane exterior domains

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Mathematical Subject classification (2010): 35Q30, 76D03, 76D05, 76D07.

Keywords: steady state Navier–Stokes equations, two–dimensional exterior Lipschitz domains, boundary–value problem, existence and asymptotic behavior of D–solutions.

Abstract

We prove that the steady–state Navier–Stokes problem in a plane Lipschitz domain Ω exterior to a bounded and simply connected set has a D–solution provided the boundary datum $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfies

$$\frac{1}{2\pi}\left|\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}\right| < 1.$$

If Ω is of class $C^{1,1}$, we can assume $\boldsymbol{a} \in W^{-1/4,4}(\partial\Omega)$. Moreover, we show that for every D–solution $(\boldsymbol{u}, p)$ of the Navier–Stokes equations it holds

$$\nabla p = o(r^{-1}), \quad \nabla_k p = O(r^{\epsilon-3/2}), \quad \nabla_k \boldsymbol{u} = O(r^{\epsilon-3/4}),$$

for all $k \in \mathbb{N} \setminus \{1\}$ and for all positive ϵ, and if the flux of $\boldsymbol{u}$ through a circumference surrounding $\complement\Omega$ is zero, then there is a constant vector $\boldsymbol{u}_0$ such that

$$\boldsymbol{u} = \boldsymbol{u}_0 + o(1).$$

1 Introduction

Let

$$\Omega = \mathbb{R}^2 \setminus \overline{\Omega'}, \tag{1}$$

with Ω' bounded and simply connected Lipschitz domain[1]. As is well–known, the steady–state Navier–Stokes problem in Ω is to find a solution $(\boldsymbol{u}, p)$ of the system [11][2]

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nabla p \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad\ \ \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} \quad\ \ \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\boldsymbol{u}$, p and $\boldsymbol{a}$ are respectively the velocity, the pressure and the boundary datum. In [26] we removed the classical zero flux condition

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0$$

for the existence of a solution of system (2)[3]. Indeed, by following the well–known approach of *invading domains* of J. Leray [21], we proved existence of a solution $(\boldsymbol{u}_\ell, p_\ell) \in D^{1,2}(\Omega) \times L^2_{\mathrm{loc}}(\overline{\Omega})$ of problem (2), we shall call Leray solution, provided $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ and

$$\frac{\kappa}{2\pi} \left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| < 1, \tag{3}$$

with

$$\kappa = \sup_{\|\boldsymbol{\varphi}\|_{D^{1,2}_\sigma(\mathbb{R}^2)} = 1} \left| \int_{\mathbb{R}^2} (\log r) \operatorname{div} (\boldsymbol{\varphi} \cdot \nabla \boldsymbol{\varphi}) \right| < +\infty. \tag{4}$$

By well–known results of D. Gilbarg & H.F. Weinberger [16] and G.P. Galdi [12] $\boldsymbol{u}_\ell$ is known to be bounded in a neighborhood of infinity $\complement C_{R_0}$ and there is a (unknown) constant vector $\boldsymbol{u}_0$ such that

$$\begin{aligned} p_\ell(x) &= o(1), \\ \boldsymbol{u}_\ell(x) &= \boldsymbol{u}_0 + o(1) \end{aligned} \tag{5}$$

[1] See Remark 4.7.

[2] As is always possible, we assume throughout the kinematical viscosity coefficient equal to 1.

[3] See [11], Ch. IX, and [12].

and

$$\nabla \boldsymbol{u}_\ell = O(r^{-3/4}\log r). \tag{6}$$

Moreover, in [11] it is proved that[4]

$$\begin{aligned} \nabla_k p_\ell(x) &= o(1),\\ \nabla_k \boldsymbol{u}_\ell(x) &= o(1), \end{aligned} \tag{7}$$

for all $k \in \mathbb{N}$.

In a recent paper [28] we improve (7) by showing that

$$\begin{aligned} \nabla_k p_\ell(x) &= O(r^{\epsilon-1/2}),\\ \nabla_k \boldsymbol{u}_\ell(x) &= O(r^{\epsilon-1/2}), \end{aligned} \tag{8}$$

for all $k \in \mathbb{N}$ and for every positive ϵ. Let us note that $(5)_1$, (7) and (8) hold for every solution $(\boldsymbol{u}, p)$ of $(2)_{1,2}$ such that [17][5]

$$\int_{\complement C_{R_0}} |\nabla \boldsymbol{u}|^2 < +\infty,$$

for some $C_{R_0} \Supset \Omega'$, *we shall call D–solution*[6]. Moreover, $(5)_1$ is replaced by the weaker one [17]

$$|\boldsymbol{u}(x)|^2 = o(\log r)$$

If $\boldsymbol{u}$ vanishes on $\partial\Omega$, C.J. Amick proved that $\boldsymbol{u}$ is bounded so that by the results of [12], [17]

$$\boldsymbol{u} = \boldsymbol{u}_0 + o(1), \tag{9}$$

with $\boldsymbol{u}_0$ constant vector. If $\boldsymbol{u}_0 \neq \mathbf{0}$ L.I. Sazonov [35] showed that $\boldsymbol{u}$ is physically reasonable in the sense of R. Finn and D.R. Smith [9], [37] so that it behaves at infinity (almost) as the solution of the Oseen problem[7]. To the best of our knowledge this is the state of the art of the problem of the existence and asymptotic behavior at infinity of a D–solution[8].

[4] We set $\nabla_k\varphi = \nabla \ldots \nabla_{k-\text{times}}\varphi$, $\nabla_1\varphi = \nabla\varphi$, $\nabla_0\varphi = \varphi$.

[5] For a D–solution (6) is replaced by $\nabla \boldsymbol{u} = O(r^{-3/4}\log^{9/8} r)$ [17].

[6] The existence of a D–solution $(\boldsymbol{u}_f, p_f)$ can be also find by a technique of H. Fujita [10] (see also [11]). Due to the lack of a uniqueness theorem we cannot compare the two solutions. However, if $\boldsymbol{u}_f$ has zero outflow through ∂C_{R_0}, then $\boldsymbol{u}_f$ is bounded (see Theorem 2).

[7] See Remark 3.3.

[8] For $\boldsymbol{u}_0 \neq \mathbf{0}$ by different approaches and under suitable smallness assumption on the data R. Finn & D.R. Smith [9] (see also [30]) and G.P. Galdi [11] proved existence of a D–solution of (2) which takes the value $\boldsymbol{u}_0$ at infinity .

In this paper we continue the study started in [26] on system (2) with a threefold main purpose:

• to prove that $\kappa \le 1$ and to get the results of [26] by weakening the hypotheses on the boundary datum; to be precise we shall only assume Ω Lipschitz and $\boldsymbol{a} \in L^q(\partial\Omega)$, $q \ge 2$, and prove existence of a D–solution of equations $(2)_{1,2}$ which takes the boundary value $\boldsymbol{a}$ in the sense of the nontangential convergence for $q > 2$; if Ω is of class $C^{1,1}$, we can assume $\boldsymbol{a} \in W^{-1/4,4}(\partial\Omega)$.

• to observe that Amick's result (9) on the boundedness of a D–solution holds under the sole hypothesis that the flux of $\boldsymbol{u}$ through ∂C_{R_0} is zero;

• starting from the results of [28] to show that for every D–solution $(\boldsymbol{u}, p)$

$$\nabla p(x) = o(r^{-1}) \tag{10}$$

and[9]

$$\begin{aligned} \nabla_k p(x) &= O(r^{\epsilon-3/2}), \\ \nabla_k \boldsymbol{u}(x) &= O(r^{\epsilon-3/4}), \end{aligned}$$

for all $k \in \mathbb{N} \setminus \{1\}$. Moreover, by means of the classical Hamel solutions we observe that (10) is sharp.

Notation – A domain (open connected set) Ω of $\mathbb{R}^2$ is said to be of class $C^{k,\alpha}$ if for every $\xi \in \partial\Omega$, there exists a neighborhood of ξ in $\partial\Omega$ which can be expressed as a graph of a function of class $C^{k,\alpha}$; for $k = 0$ and $\alpha = 1$ Ω is said to be Lipschitz. We shall use a standard vector notation, as in [11]; $\{o, (\boldsymbol{e}_1, \boldsymbol{e}_2)\}$ is a cartesian reference frame of $\mathbb{R}^2$ with origin o and $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ orthonormal basis of $\mathbb{R}^2$; $\{o, (\boldsymbol{e}_r, \boldsymbol{e}_\theta)\}$ is the polar coordinate system with origin at o; $x = (x_1, x_2) = (r, \theta)$ denotes the generic point of $\mathbb{R}^2$, with $r = |\boldsymbol{x}|$, $\boldsymbol{x} = x - o = r\boldsymbol{e}_r$; if $\boldsymbol{u}$ is a vector field in $\mathbb{R}^2$, by (u_1, u_2) and (u_r, u_θ) we denote the cartesian and polar components of $\boldsymbol{u}$ respectively, and we set $(\nabla\boldsymbol{u})_{ij} = \partial u_j/\partial x_i$, $\partial_r \boldsymbol{u} = \boldsymbol{e}_r \cdot \nabla\boldsymbol{u}$, $\partial_\theta \boldsymbol{u} = \boldsymbol{e}_\theta \cdot \nabla\boldsymbol{u}$, $\nabla^\perp = (-\partial_2, \partial_1)$. C_R is the disk of radius R centered at o; also, we set $T_R = C_{2R} \setminus \overline{C}_R$, $\Omega_R = \Omega \cap C_R$; if Ω_1 and Ω_2 are two domains, $\Omega_1 \Subset \Omega_2$ means that $\overline{\Omega_1} \subset \Omega_2$; if Ω is the exterior domain (1) we denote by R_0 a positive constant such that $\Omega' \Subset C_{R_0}$; the symbol c will be reserved to denote a positive constant whose numerical value is unessential to our purposes. We use a standard notation to denote (scalar, vector or second–order tensor) function spaces, as in [11] and, in particular $D^{k,q}(\Omega)$ denotes the Banach space of all fields $\varphi \in L^1_{\rm loc}(\Omega)$ such that $\|\nabla_k \varphi\|_{L^q(\Omega)} < +\infty$; $D^{k,2}_\delta(\Omega) = \{\varphi \in L^1_{\rm loc}(\Omega) : \|\sqrt{\delta}\nabla_k\varphi\|_{L^2(\Omega)} < +\infty\}$, where $\delta = \delta(x)$ is a function equal to the distance of x from $\partial\Omega$ in a neighborhood of $\partial\Omega$ and to 1 in C_{R_0}; $\mathcal{H}^q$ $(q > 0)$ stands for the Hardy space in $\mathbb{R}^2$ [39]. The symbol V_σ, where $V(\subset L^1_{\rm loc}(\Omega))$ stands for the subset of V of all vector fields u such that

[9] As will appear clear from the proof the quantity r^ϵ can be replaced by a suitable power of $\log r$ depending on k.

$\int_\Omega u\cdot\nabla\varphi = 0$, for all $\varphi \in C_0^\infty(\Omega)$. Let φ be a function in Ω. Let $\{\gamma(\xi)\}_{\xi\in\partial\Omega}$ be a family of circular finite (not empty) triangles with vertex on $\partial\Omega$ such that $\gamma(\xi)\setminus\{\xi\}\subset\Omega$[10]; $\varphi(x)$ is said to converge nontangentially at the boundary if

$$\varphi(\xi) = \lim_{\substack{x\to\xi \\ (x\in\gamma(\xi))}} \varphi(x) \Leftrightarrow \varphi(x) \xrightarrow{\mathrm{nt}} \varphi(\xi)$$

for almost all $\xi\in\partial\Omega$. The (Landau) symbols $f(x) = o(g(r))$ and $f(x) = O(g(r))$ ($g > 0$) mean respectively that $\lim_{r\to+\infty}(f/g) = 0$ and f/g is bounded in a neighborhood of infinity. If $\varphi \in L^1(\Omega)$ [or $\varphi \in \partial\Omega$] we use the symbol

$$\int_\Omega \varphi \quad \left[\int_{\partial\Omega}\varphi\right]$$

to denote the integral of φ over Ω [on $\partial\Omega$].

2 Some Lemmas

Throughout the paper we shall consider the domain Ω defined by (1) and, as is always possible, we assume that $C_1 \Subset \Omega'$.

Let us start by recalling some well–known results concerning the Stokes problem

$$\begin{aligned} \Delta\boldsymbol{u} &= \nabla p && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{a} && \text{on } \partial\Omega, \end{aligned} \tag{11}$$

we shall use in the sequel.

It is well–known that if $\boldsymbol{a}\in L^2(\partial\Omega)$, then (11) has an analytical D–solution in Ω [8], [33], [34] expressed by

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\sigma}, \tag{12}$$

with

$$\boldsymbol{\sigma}(x) = -\frac{\boldsymbol{e}_r}{2\pi r}\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}$$

and $\boldsymbol{n}$ outward (with respect to Ω) unit normal to $\partial\Omega$, such that $\boldsymbol{u}$ tends nontangentially to $\boldsymbol{a}$ and

$$\int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{n} = 0. \tag{13}$$

[10] As is well–known, since Ω is Lipschitz such a family of triangles certainly exists.

It is unique in the class of the so–called very weak solutions [27], [34]. Moreover, there is a constant vector $\boldsymbol{u}_0$[11] such that [24], [33], [34]

$$\begin{aligned}\nabla_k(\boldsymbol{u}-\boldsymbol{u}_0)&=O(r^{-1-k}),\\ \nabla_k p&=O(r^{-2-k})\end{aligned}\tag{14}$$

and

$$\int_\Omega \delta(|\nabla\boldsymbol{u}|^2+|p|^2)\le c\int_{\partial\Omega}|\boldsymbol{a}|^2.\tag{15}$$

Since $D^{1,2}_\delta(\Omega_R)\hookrightarrow W^{1/2,2}(\Omega_R)\hookrightarrow L^4(\Omega_R)$ [18], by $(14)_1$ we have in particular that $\boldsymbol{u}-\boldsymbol{u}_0\in L^4(\Omega)$. Moreover, it holds (see, *e.g.*, [24], [33], [34], [36])

(i) if $\boldsymbol{a}\in L^q(\partial\Omega)$, $q\in[2,+\infty]$, then $\boldsymbol{u}\in W^{1/q,q}_{\rm loc}(\overline{\Omega})$.

There are two positive scalars $\mu_0(<1)$ and ε depending only on $\partial\Omega$ such that

(ii) if $\boldsymbol{a}\in C^{0,\mu}(\partial\Omega)$, $\mu\in[0,\mu_0)$, then $\boldsymbol{u}\in C^{0,\mu}_{\rm loc}(\overline{\Omega})$; if Ω is of class C^1, we can take $\mu_0=1$;

(iii) if $\boldsymbol{a}\in W^{1-1/q,q}(\partial\Omega)$, $q\in[2,2+\varepsilon)$, then $(\boldsymbol{u},p)\in W^{1,q}_{\rm loc}(\overline{\Omega})\times L^q_{\rm loc}(\overline{\Omega})$; if Ω is of class C^1, we can take $q\in(1+\infty)$;

(iv) if $\boldsymbol{a}\in W^{1,q}(\partial\Omega)$, $q\in(2-\epsilon,2+\varepsilon)$, then $(\boldsymbol{u},p)\in W^{1+1/q,q}_{\rm loc}(\overline{\Omega})\times W^{1/q,q}_{\rm loc}(\overline{\Omega})$; if Ω is of class C^1, we can take $q\in(1+\infty)$. Moreover, if $\boldsymbol{a}\in W^{1,2}(\partial\Omega$, then

$$\int_\Omega\big[|\nabla\boldsymbol{u}|^2+|p|^2+\delta(|\nabla\boldsymbol{u}|^2+|p|^2)\big]\le c\int_{\partial\Omega}(|\boldsymbol{a}|^2+|\nabla\boldsymbol{a}|^2).$$

The above results allow us to prove

Lemma 1. *Let Ω be a Lipschitz exterior domain of $\mathbb{R}^2$. If $\boldsymbol{a}\in L^2(\partial\Omega)$, then there is a field $\boldsymbol{h}\in C^\infty_\sigma(\Omega)\cap D^{1,2}_\delta(\Omega)$ which tends nontangentially to $\boldsymbol{a}$ on $\partial\Omega$, vanishes outside a disk and satisfies*

$$\|\boldsymbol{h}\|_{L^4(\Omega)}\le c\|\boldsymbol{h}\|_{D^{1,2}_\delta(\Omega)}\le c\|\boldsymbol{a}\|_{L^2(\partial\Omega)}.\tag{16}$$

Moreover, if $\boldsymbol{a}$ is more regular, then also $\boldsymbol{h}$ is more regular according to (i)–(iv).

[11] $\boldsymbol{u}_0$ is determined by $\boldsymbol{a}$ through well–known compatibility conditions (see, *e.g.*, [11], [27], [33], [34]).

Proof – Let g be a C^∞ cut–off function in $\mathbb{R}^2$, equal to 1 in $C_{\bar R}$ and to zero outside $C_{2\bar R}$ with $\bar R > R_0$. Since by (13)

$$\int_{T_{\bar R}} \operatorname{div}(g\boldsymbol{v}) = 0,$$

the problem

$$\operatorname{div}\boldsymbol{\omega} + \operatorname{div}(g\boldsymbol{v}) = 0 \quad \text{in } T_{\bar R}$$

admits a solution $\boldsymbol{\omega} \in C_0^\infty(T_{\bar R})$ [25] (see also [11] Ch.III). It is clear that the field

$$\boldsymbol{h}(x) = \boldsymbol{\zeta}(x) + \boldsymbol{\sigma}, \qquad \boldsymbol{\zeta} = \begin{cases} \boldsymbol{v}, & \text{in } \Omega_{\bar R}, \\ \boldsymbol{\omega} + g\boldsymbol{v}, & \text{in } T_{\bar R}, \\ \boldsymbol{0}, & \text{in } \mathbb{R}^2 \setminus \Omega_{2\bar R}, \end{cases} \tag{17}$$

satisfies all the properties stated in the Lemma. □

If Ω is of class $C^{1,1}$ and $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$ $(q > 1)$[12], then (11) admits the solution (12) where $\boldsymbol{v}$ is a simple layer potential (plus a constant vector $\boldsymbol{u}_0$) with a density in $W^{-1-1/q,q}(\partial\Omega)$ [6], [34]. The boundary datum is taken in the sense of the unique continuous extension map from $W^{-1-1/q,q}(\partial\Omega)$ into $W^{-1/q,q}(\partial\Omega)$ of the trace operator of the classical simple layer potential from $W^{1/q,q}(\partial\Omega)$ to $W^{1+1/q,q}(\partial\Omega)$ [6] . Moreover, $\boldsymbol{u} \in L^q_{\text{loc}}(\overline{\Omega})$ satisfies (14). Therefore, by proceeding as we did in the proof of Lemma 1 and taking also into account the regularity properties of the classical layer potentials [22], [34], we have

Lemma 2. *Let Ω be a exterior domain of $\mathbb{R}^2$ of class $C^{1,1}$. If $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$, $q \geq 4$, then there is a divergence free extension $\boldsymbol{h} \in C_\sigma^\infty(\Omega) \cap L^q(\Omega)$ of $\boldsymbol{a}$ in Ω, expressed by (17), which satisfies*

$$\|\boldsymbol{h}\|_{L^q(\Omega)} \leq c\|\boldsymbol{a}\|_{W^{-1/q,q}(\partial\Omega)}. \tag{18}$$

Moreover,

- *if $\boldsymbol{a} \in C^{1,\mu}(\partial\Omega)$, $\mu < 1$, then $\boldsymbol{h} \in C^{1,\mu}(\overline{\Omega})$;*
- *if Ω is of class C^k, $k \geq 2$ and $\boldsymbol{a} \in W^{k-1/q,q}(\partial\Omega)$, $\mu < 1$, then $\boldsymbol{h} \in W^{k+2,q}(\Omega)$.*

The following elementary but basic Lemma was first proved in [17]. We give a simple proof of a slight generalization.

[12] $W^{-1/q,q}(\partial\Omega)$ is the dual space of $W^{1-1/q',q'}(\partial\Omega)$.

Lemma 3. *Let* $\boldsymbol{w}, \boldsymbol{z} \in W^{1,2}_{\sigma}(C_R)$. *Then*

$$\left| \int_{C_R} \boldsymbol{w} \cdot \nabla \boldsymbol{z} \cdot \frac{\boldsymbol{e}_r}{r} \right| \le \left\{ \int_{C_R} |\nabla \boldsymbol{w}|^2 \int_{C_R} |\nabla \boldsymbol{z}|^2 \right\}^{1/2}. \tag{19}$$

Proof – Set

$$\bar{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r,\theta).$$

Since

$$\int_0^{2\pi} \bar{w}_1 \partial_\theta z_2 = \int_0^{2\pi} \bar{w}_2 \partial_\theta z_1 = 0,$$

a simple computation yields [27]

$$\int_{C_R} \boldsymbol{w} \cdot \nabla \boldsymbol{z} \cdot \frac{\boldsymbol{e}_r}{r} = \int_0^R \frac{1}{\rho} \int_0^{2\pi} \big[(w_1 - \bar{w}_1)\partial_\theta z_2 - (w_2 - \bar{w}_2)\partial_\theta z_1\big]. \tag{20}$$

From Schwarz's, Wirtinger's and Cauchy's inequalities we have

$$\begin{aligned} \left| \int_0^{2\pi} (w_1 - \bar{w}_1)\partial_\theta z_2 \right| &\le \left\{ \int_0^{2\pi} |w_1 - \bar{w}_1|^2 \int_0^{2\pi} |\partial_\theta z_2|^2 \right\}^{1/2} \\ &\le \left\{ \int_0^{2\pi} |\partial_\theta w_1|^2 \int_0^{2\pi} |\partial_\theta z_2|^2 \right\}^{1/2}, \\ \left| \int_0^{2\pi} (w_2 - \bar{w}_2)\partial_\theta z_1 \right| &\le \left\{ \int_0^{2\pi} |\partial_\theta w_2|^2 \int_0^{2\pi} |\partial_\theta z_1|^2 \right\}^{1/2}. \end{aligned}$$

Therefore, (19) follows from (20), taking into account that $|\partial_\theta \boldsymbol{w}| \le r|\nabla \boldsymbol{w}|$. $\square$

Note that if $\boldsymbol{w}$, $\boldsymbol{z} \in D^{1,2}_{\sigma}(\mathbb{R}^2)$, letting $R \to +\infty$ in (19) yields

$$\left| \int_{\mathbb{R}^2} \boldsymbol{w} \cdot \nabla \boldsymbol{z} \cdot \frac{\boldsymbol{e}_r}{r} \right| \le \left\{ \int_{\mathbb{R}^2} |\nabla \boldsymbol{w}|^2 \int_{\mathbb{R}^2} |\nabla \boldsymbol{z}|^2 \right\}^{1/2}. \tag{21}$$

If $\boldsymbol{w}, \boldsymbol{z} \in W^{1,2}_{\sigma,0}(\Omega)$, then the zero extensions of $\boldsymbol{w}$ and $\boldsymbol{z}$ belongs to $D^{1,2}_{\sigma}(\mathbb{R}^2)$. Therefore, from (21) it follows

$$\left| \int_{\Omega} \boldsymbol{w} \cdot \nabla \boldsymbol{z} \cdot \frac{\boldsymbol{e}_r}{r} \right| \le \left\{ \int_{\Omega} |\nabla \boldsymbol{w}|^2 \int_{\Omega} |\nabla \boldsymbol{z}|^2 \right\}^{1/2}. \tag{22}$$

Lemma 3 allows to quickly prove

Theorem 1. *It holds*

$$\sup_{\|\boldsymbol{\varphi}\|_{D^{1,2}_\sigma(\mathbb{R}^2)}=1} \left| \int_{\mathbb{R}^2} (\log|\boldsymbol{x}|) \mathrm{div}\,(\boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi}) \right| \le 1. \tag{23}$$

Proof – It is sufficient to prove (23) in $C^\infty_{\sigma,0}(\mathbb{R}^2)$. In this case

$$\left| \int_{\mathbb{R}^2} (\log|\boldsymbol{x}|) \mathrm{div}\,(\boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi}) \right| = \left| \int_{\mathbb{R}^2} \boldsymbol{\varphi}\cdot\nabla\boldsymbol{\varphi}\cdot \frac{\boldsymbol{e}_r}{r} \right|$$

and (23) follows from Lemma 3. □

Lemma 4. *Let* $(\boldsymbol{u}, p)$ *be a a solution to* $(2)_{1,2}$*. Then for all* $k \in \mathbb{N}_0$ *and for all* $C_1(x) \Subset \Omega$

$$|\nabla_k p(x)| \le c \left\{ \int_0^{2\pi} |\nabla_k p|(|\boldsymbol{x}|,\theta) + \sum_{j=1}^{k+1} \|\nabla_j \boldsymbol{u}\|^2_{L^2(C_1(x))} \right\}. \tag{24}$$

Proof – We follow [12] (Lemma 3.10). Setting $\boldsymbol{u}_1 = \nabla\boldsymbol{u}$ and $p_1 = \nabla p$, the pair $(\boldsymbol{u}_1, p_1)$ is a solution of the equations

$$\begin{aligned} \Delta\boldsymbol{u}_1 - \boldsymbol{u}_1\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}_1 &= \nabla p_1, \\ \mathrm{div}\,\boldsymbol{u}_1 &= 0. \end{aligned} \tag{25}$$

Let $x = (r,\theta)$ and let (r',θ') be a polar coordinate system centered at x. Multiplying $(25)_1$ scalarly by $\boldsymbol{x}'/{r'}^2$ and integrating over $C_1(x)$, we have

$$p_1(r,\theta) = p_1(r',\theta') + \int_{C_1(x)} [\boldsymbol{u}_1\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_1]\cdot\frac{\boldsymbol{x}'}{{r'}^2}$$

Hence, making use of Lemma 3, it follows

$$\begin{aligned} &|p_1(r,\theta)| \le \frac{1}{2\pi}\left\{ \int_0^{2\pi} |p_1(r',\theta')| + \|\nabla\boldsymbol{u}\|_{L^2(C_1(x))}\|\nabla\boldsymbol{u}_1\|_{L^2(C_1(x))} \right\} \\ &\int_0^{2\pi} |p_1(r',\theta')| \le 2\pi\left\{ |p_1(r,\theta)| + \|\nabla\boldsymbol{u}\|_{L^2(C_1(x))}\|\nabla\boldsymbol{u}_1\|_{L^2(C_1(x))}.\right\} \end{aligned} \tag{26}$$

Multiplying $(26)_2$ by r' and integrating over $r' \in [0,1]$ and $\theta \in [0,2\pi]$ show that

$$\|p_1\|_{L^1(C_1(x))} \le c\left\{ \int_0^{2\pi} |p_1(r,\theta)| + \|\nabla\boldsymbol{u}\|_{L^2(C_1(x))}\|\nabla\boldsymbol{u}_1\|_{L^2(C_1(x))} \right\}. \tag{27}$$

Moreover, multiplying $(26)_1$ by r' and integrating over $r' \in [0,1]$ and $\theta' \in [0,2\pi]$ yield

$$|p_1(x)| \leq c\left\{\|p_1\|_{L^1(C_1(x))} + \|\nabla \boldsymbol{u}\|_{L^2(C_1(x))}\|\nabla \boldsymbol{u}_1\|_{L^2(C_1(x))}\right\}. \tag{28}$$

Therefore, putting together (27)–(28) and using Cauchy's inequality we find

$$\begin{aligned}
|p_1(x)| &\leq c\left\{\int_0^{2\pi} |p_1(r,\theta)| + \|\nabla \boldsymbol{u}\|_{L^2(C_1(x))}\|\nabla \boldsymbol{u}_1\|_{L^2(C_1(x))}\right\} \\
&\leq c\left\{\int_0^{2\pi} |p_1(r,\theta)| + \|\nabla \boldsymbol{u}\|^2_{L^2(C_1(x))} + \|\nabla \boldsymbol{u}_1\|^2_{L^2(C_1(x))}\right\}.
\end{aligned}$$

and (24) is proved for $k = 1$. The proof for $k = 0$ follows the same steps. Iterating such a procedure as many times as we need, we then prove (24). $\square$

Lemma 5. [11] *Let*

$$v(x) = \int_{\mathbb{R}^2} \frac{1}{|x-y|^\lambda |y|^\mu} \mathrm{d}a_y,$$

with $\lambda < 2$, $\mu < 2$. *If* $\lambda + \mu > 2$, *then*

$$v(x) = cr^{2-\lambda-\mu},$$

for a suitable constant $c = c(\lambda,\mu)$.

Lemma 6. [39] *If* $f \in \mathcal{H}^1$, *then the problem*

$$\begin{aligned}
\Delta p &= f \quad \text{in } \mathbb{R}^2, \\
\lim_{x\to\infty} p(x) &= 0
\end{aligned}$$

admits the unique solution

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(y)\log|x-y|\mathrm{d}a_y \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2).$$

Lemma 7. [5] *If* $\boldsymbol{u} \in D^{1,2}_\sigma(\mathbb{R}^2)$, *then* $\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^{\mathrm{T}} \in \mathcal{H}^1$.

3 Asymptotic behavior of D–solutions

Let us recall that by D–solution we mean an analytical pair $(\boldsymbol{u}, p)$ which satisfies equations $(2)_{1,2}$ and

$$\int_{\complement C_{R_0}} |\nabla \boldsymbol{u}|^2 < +\infty, \tag{29}$$

for some $C_{R_0} \Supset \Omega'$.

We deal now with the asymptotic properties of a D–solution. To this end we need the following classical results of D. Gilbarg and H.F. Weinberger [17].

Lemma 8. *If $(\boldsymbol{u}, p)$ is a D–solution, then*

$$\lim_{x \to +\infty} p(x) = 0 \tag{30}$$

and

$$\boldsymbol{u} = o(\sqrt{\log r}). \tag{31}$$

Moreover

$$\nabla \boldsymbol{u} = O(r^{-3/4} \log^{9/8} r). \tag{32}$$

Also, it holds [28]

Lemma 9. *If $(\boldsymbol{u}, p)$ is a D–solution, then*

$$\begin{aligned} \nabla_k p(x) &= O(r^{\epsilon - 1/2}), \\ \nabla_k \boldsymbol{u}(x) &= O(r^{\epsilon - 1/2}), \end{aligned} \tag{33}$$

for all $k \in \mathbb{N}$.

The following theorem extends to more general boundary data a classical result of C.J. Amick [2], D. Gilbarg and H.F. Weinberger [17] and G.P. Galdi [12].

Theorem 2. *Let $(\boldsymbol{u}, p)$ be a D–solution. If*

$$\int_{\partial C_{R_0}} u_{R_0} = 0, \tag{34}$$

then there is a constant vector $\boldsymbol{u}_0$ such that

$$\boldsymbol{u} = \boldsymbol{u}_0 + o(1). \tag{35}$$

Proof – From $(2)_2$ and (34) it follows that there is a regular function ψ such that $\boldsymbol{u} = \nabla^{\perp} \psi$. Therefore, we can repeat the argument of Section 2.1 of [2] to see that there is a curve connecting a point of ∂C_{R_0} to infinity along which the Bernoulli function $\Phi = p + \frac{1}{2} |\boldsymbol{u}|^2$ is monotone decreasing ((b) of Theorem

11) and this is sufficient to assert that $\boldsymbol{u}$ is bounded (Theorem 12). Hence by Theorem 4 of [17] there is a constant vector $\boldsymbol{u}_0$ such that

$$\lim_{r\to+\infty}\int_0^{2\pi}|\boldsymbol{u}(r,\theta)-\boldsymbol{u}_0|^2=0. \tag{36}$$

Since $\nabla\boldsymbol{u}\in L^q(\complement C_{R_0})$ for all $q\geq 2$ (see, *e.g.*, [11] Lemma X.3.2), (36) implies (35) by virtue of Lemma 3.10 of [12]. $\square$

The following theorem concerns the asymptotic behavior of the derivatives of a D–solution.

Theorem 3. *If $(\boldsymbol{u},p)$ is a D–solution, then*

$$p(x)\in D^{1,2}(\complement C_{R_0}), \tag{37}$$

$$\nabla p(x)=o(r^{-1}) \tag{38}$$

and

$$\begin{aligned}\nabla_{k+1}p(x)&=O(r^{\epsilon-3/2}),\\ \nabla_k\boldsymbol{u}&=O(r^{\epsilon-3/4}),\end{aligned} \tag{39}$$

for every positive ϵ and for every $k\in\mathbb{N}$. Moreover, if $\boldsymbol{u}$ satisfies (34), then $p\in D^{1,2}(\complement C_{R_0})$.

Proof of (37), (38).

Let
$$\boldsymbol{\gamma}(x)=\frac{\boldsymbol{e}_r}{r}\int_{\partial C_{R_0}}u_{R_0}$$
and set
$$\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{\gamma}.$$
Let g be a regular cut–off function in $\mathbb{R}^2$, vanishing in $C_{\bar{R}}$ and equal to 1 outside $C_{2\bar{R}}$, with $\bar{R}\gg R_0$. Since
$$\int_{\partial C_{\bar{R}}}\boldsymbol{v}\cdot\boldsymbol{n}=0,$$
the problem
$$\operatorname{div}\boldsymbol{h}+\operatorname{div}(g\boldsymbol{v})=0\quad\text{in } T_{\bar{R}}$$

has a solution $\boldsymbol{h} \in C_0^\infty(T_{\bar{R}})$ [11]. From $(2)_{1,2}$ it follows that the function $Q = g^2 p$ is a solution of the equation

$$\Delta Q + \sum_{i=1}^{3} \operatorname{div} \boldsymbol{h}_i + \varphi = 0 \quad \text{in } \mathbb{R}^2, \tag{40}$$

where

$$\varphi \in C_0^\infty(T_{\bar{R}})$$

and

$$\begin{aligned}
\boldsymbol{h}_1 &= (g\boldsymbol{v} + \boldsymbol{h}) \cdot \nabla(g\boldsymbol{v} + \boldsymbol{h}),\\
\boldsymbol{h}_2 &= 2g\boldsymbol{v} \cdot \nabla(g\boldsymbol{\gamma}),\\
\boldsymbol{h}_3 &= g\boldsymbol{\gamma} \cdot \nabla(g\boldsymbol{\gamma}).
\end{aligned}$$

By virtue of (30) equation (40) has a unique solution Q which by Lemma 5 is expressed by

$$\begin{aligned}
2\pi Q(x) &= -\sum_{i=1}^{3} \int_{\mathbb{R}^2} (\log|x-y|) \operatorname{div} \boldsymbol{h}_i(y) \mathrm{d}a_y - \int_{\mathbb{R}^2} \varphi(y) \log|x-y| \mathrm{d}a_y\\
&= \sum_{1=1}^{4} Q_i.
\end{aligned} \tag{41}$$

By Lemma 7 $\operatorname{div} \boldsymbol{h}_1 \in \mathcal{H}^1$ so that Lemma 6 implies that $Q_1 \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2)$. Hence it follows in particular that

$$\lim_{x \to +\infty} Q_1(x) \in \mathbb{R}. \tag{42}$$

Now, letting $R \to +\infty$ in the relation

$$\begin{aligned}
\int_{C_R} (\log|x-y|) \operatorname{div}[(g\boldsymbol{v}) \cdot \nabla(g\boldsymbol{\gamma})] \mathrm{d}a_y &= \int_{\partial C_R} (\log|x-\zeta|)(\boldsymbol{v} \cdot \nabla\boldsymbol{\gamma} \cdot \boldsymbol{e}_R)(\zeta) \mathrm{d}s_\zeta\\
&\quad + \int_{C_R} \frac{[g\boldsymbol{v} \cdot \nabla(g\boldsymbol{\gamma})](y) \cdot (x-y)}{|x-y|^2} \mathrm{d}a_y,\\
\int_{C_R} (\log|x-y|) \operatorname{div}[(g\boldsymbol{\gamma}) \cdot \nabla(g\boldsymbol{\gamma})] \mathrm{d}a_y &= \int_{\partial C_R} (\log|x-\zeta|)(\boldsymbol{\gamma} \cdot \nabla\boldsymbol{\gamma} \cdot \boldsymbol{e}_R)(\zeta) \mathrm{d}s_\zeta\\
&\quad + \int_{C_R} \frac{[g\boldsymbol{\gamma} \cdot \nabla(g\boldsymbol{\gamma})](y) \cdot (x-y) \mathrm{d}a_y}{|x-y|^2}
\end{aligned}$$

and taking into account the behavior at infinity of $\boldsymbol{v}$ and $\boldsymbol{\gamma}$, we have

$$
\begin{aligned}
Q_2(x) &= -2\int_{\mathbb{R}^2} \frac{[g\boldsymbol{v}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y,\\
Q_3(x) &= -\int_{\mathbb{R}^2} \frac{[g\boldsymbol{\gamma}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y),}{|x-y|^2}\mathrm{d}a_y.
\end{aligned} \tag{43}
$$

Now, for large $|\boldsymbol{x}|$

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{[g\boldsymbol{v}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y &= \int_{\mathbb{R}^2\setminus C_1(x)} \frac{[g\boldsymbol{v}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y\\
&\quad+\int_{C_1(x)} \frac{(\boldsymbol{v}\cdot\nabla\boldsymbol{\gamma})(y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y\\
&= \int_{\mathbb{R}^2\setminus C_1(x)} \frac{[g\boldsymbol{v}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y\\
&\quad-\int_{\partial C_1(x)} (\log|x-y|)(\boldsymbol{v}\cdot\nabla\boldsymbol{\gamma})(\zeta)\cdot\boldsymbol{n}(\zeta)\mathrm{d}a_\zeta\\
&\quad+\int_{C_1(x)} (\log|x-y|)(\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{\gamma}^{\mathrm{T}})(y)\mathrm{d}a_y.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\nabla Q_2(x) &= -2\nabla\int_{\mathbb{R}^2\setminus C_1(x)} \frac{[g\boldsymbol{v}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y\\
&\quad+2\int_{\partial C_1(x)} \frac{(x-y)(\boldsymbol{v}\cdot\nabla\boldsymbol{\gamma})(\zeta)\cdot\boldsymbol{n}(\zeta)}{|x-y|^2}\mathrm{d}a_\zeta\\
&\quad-2\int_{C_1(x)} \frac{(x-y)(\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{\gamma}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y.
\end{aligned} \tag{44}
$$

Likewise, since

$$
\begin{aligned}
\nabla Q_3(x) &= -2\nabla\int_{\mathbb{R}^2\setminus C_1(x)} \frac{[g\boldsymbol{\gamma}\cdot\nabla(g\boldsymbol{\gamma})](y)\cdot(x-y)}{|x-y|^2}\mathrm{d}a_y\\
&\quad+2\int_{\partial C_1(x)} \frac{(x-y)(\boldsymbol{\gamma}\cdot\nabla\boldsymbol{\gamma})(\zeta)\cdot\boldsymbol{n}(\zeta)}{|x-y|^2}\mathrm{d}a_\zeta\\
&\quad-2\int_{C_1(x)} \frac{(x-y)(\nabla\boldsymbol{\gamma}\cdot\nabla\boldsymbol{\gamma}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y,
\end{aligned} \tag{45}
$$

taking into account the asymptotic properties of $\boldsymbol{v}$, $\nabla \boldsymbol{v}$ and $\boldsymbol{\gamma}$, (43), (44) and (45) imply

$$Q_2(x),\ Q_3(x) = O(r^{\epsilon-1}) \tag{46}$$

and

$$\nabla Q_2(x),\ \nabla Q_3(x) = O(r^{\epsilon-2}), \tag{47}$$

for all positive ϵ. By virtue of (30)[13]

$$\int_{\mathbb{R}^2} \varphi = 0$$

so that

$$\nabla_k Q_4(x) = O(r^{-1-k}), \tag{48}$$

for all $k \in \mathbb{N}$, and (37) is proved. By the basic calculus and $(7)_1$

$$\int_0^{2\pi} |\nabla Q_1|(R,\theta) = \left| \int_R^{+\infty} \partial_r \nabla Q_1 \right| \le \frac{c}{R} \int_{\complement S_R} |\nabla_2 Q_1|.$$

Hence

$$\lim_{R\to+\infty} \left\{ R \int_0^{2\pi} |\nabla Q_1|(R,\theta) \right\} = 0. \tag{49}$$

Then, (38) follows from Lemma 4, taking into account (47), (48), (49) and that $p(x) = Q(x)$ for large $|\boldsymbol{x}|$.

PROOF OF (39).

Since

$$\Delta p = -\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^{\mathrm{T}} \quad \text{in } \complement C_{R_0},$$

writing the Stokes formula in $S_R \cap \complement C_{R_0}$, taking the gradient, letting $R \to +\infty$

[13] Otherwise $p(x)$ behaves at infinity as $\log r$.

and taking into account (30), $(33)_1$, we have

$$
\begin{aligned}
2\pi\nabla p(x) &= -\int_{\partial C_{R_0}} \frac{(x-\zeta)\partial_r p(\zeta)}{|x-\zeta|^2}\mathrm{d}s_\zeta + \nabla\int_{\partial C_{R_0}} \frac{p(\zeta)(x-\zeta)\cdot \boldsymbol{e}_{R_0}}{|x-\zeta|^2}\mathrm{d}s_\zeta\\
&\quad -\int_{\complement C_{R_0}} \frac{(x-y)(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y\\
&= -\int_{\complement C_{R_0}} \frac{(x-y)(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y + \psi(x)\\
&= -\int_{\complement C_{R_0}\setminus C_1(x)} \frac{(x-y)(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y\\
&\quad +\int_{\partial C_1(x)} (\log|x-\zeta|)(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(\zeta)\boldsymbol{n}(\zeta)\mathrm{d}s_\zeta\\
&\quad -\int_{C_1(x)} (\log|x-y|)\nabla(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)\mathrm{d}a_y + \psi(x),
\end{aligned}
\tag{50}
$$

with

$$
\nabla_k\psi(x) = O(r^{-1-k}).
$$

Hence, taking the gradient, it follows

$$
2\pi\nabla_2 p(x) = \sum_{i=1}^{3}\mathcal{J}_i + O(r^{-2}),
\tag{51}
$$

where

$$
\begin{aligned}
\mathcal{J}_1 &= -\nabla\int_{\complement C_{R_0}\setminus C_1(x)} \frac{(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)(x-y)}{|x-y|^2}\mathrm{d}a_y,\\
\mathcal{J}_2 &= -\int_{C_1(x)} \frac{(x-y)\otimes\nabla(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)}{|x-y|^2}\mathrm{d}a_y,\\
\mathcal{J}_3 &= \nabla\int_{\partial C_1(x)} (\log|x-\zeta|)(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(\zeta)(x-\zeta)\boldsymbol{n}(\zeta)\mathrm{d}s_\zeta.
\end{aligned}
$$

Setting $|\boldsymbol{x}| = R$ $(R > R_0)$ and after portioning $\complement C_{R_0}$ into $\complement C_{R_0}\cap C_{R/2}$, $\complement C_{R/2}$ and taking into account (32), we get

$$
|\mathcal{J}_1(x)| \le \frac{c}{R^2}\int_{\complement C_{R_0}\cap C_R} |\nabla\boldsymbol{u}|^2 + c\int_{\complement C_R} r^{\epsilon-7/2} \le c|\boldsymbol{x}|^{\epsilon-3/2}
$$

for some positive constant c independent of R. Also, by (32)–(33) it is readily seen that $\mathcal{J}_2(x), \mathcal{J}_3(x) = O(r^{\epsilon-3/2})$. Hence (39) follows for $k = 2$. The proof of $(39)_1$ for general k is obtained by iterating the above argument.

The proof of $(39)_2$ follows the above steps, taking into account that by (31), (32) and (38)

$$\Delta \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = O(r^{-3/4} \log^{13/8} r). \qquad \square$$

Remark 3.1 - Note that under assumption (34) in (41) $Q_2 = Q_3 = 0$ so that $p \in D^{2,1}(\complement C_{R_0})$ [28]. $\diamond$

Remark 3.2 - The first basic result in [17] assures that

$$\nabla_2 \boldsymbol{u} \in L^2(\complement C_{R_0}). \tag{52}$$

Hence, taking into account (31) and $(2)_1$, it follows that $\nabla p/\sqrt{\log r} \in L^2(\complement C_{R_0})$. This is sufficient to say that (40) has the unique solution (41). Therefore, (30) follows from (42), (46) and (48). In this way we gave an alternative proof of (30) based only on (52). Note that from (52) and (38) it follows that $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^2(\complement C_{R_0})$. $\diamond$

Remark 3.3 - The asymptotic results in Theorem 3 are new in the case where $\boldsymbol{u}$ is unbounded[14] or tends to zero at large distance. Indeed, if $\boldsymbol{u}$ tends to $\boldsymbol{e}_1$ (say) at infinity, L.I. Sazonov [35] showed that $(\boldsymbol{u}, p)$ is physically meaningful in the sense of R. Finn an D.R. Smith [9], [37]. Therefore the solution enjoys the following summability properties (see, *e.g.*, [11] Ch. X)

$$\begin{aligned}
&u_1 - 1 \in L^q(\Omega),\ u_2 \in L^{q-1}(\Omega),\ \ p \in L^{q-1}(\complement C_{R_0}), \quad \forall\, q > 3,\\
&\partial_2 u_1 \in L^s(\complement C_{R_0}), \quad \forall\, s > 3/2,\\
&\partial_1 u_1, \nabla u_2,\ \nabla_2 \boldsymbol{u},\ \nabla p \in L^t(\complement C_{R_0}), \quad \forall\, t > 1.
\end{aligned} \tag{53}$$

We can say just a little bit more about the second derivatives of p. Assuming for simplicity $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, the solution p of the equation

$$\Delta p + \operatorname{div}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0$$

can be written

$$p(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log |x - y|) \operatorname{div}(\boldsymbol{u} \cdot \nabla \boldsymbol{u})(y) \mathrm{da}_y + \varpi(x) = Q(x) + \varpi(x),$$

[14] By Theorem 2 this could happens only if $\int_{\partial C_{R_0}} u_r \neq 0$.

where $\varpi(x)$ is a simple layer harmonic potential with a density having zero integral mean over $\partial\Omega$. By (53) and Theorem II.2 of [5] $\operatorname{div}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\in\mathcal{H}^t$, for all $t>2/3$. Therefore, by well–known results about singular integrals (see, *e.g*, [39] p. 136) we have that

$$\nabla_2 p(x)=\nabla_2 Q(x)+O(r^{-3}),$$

with $\nabla_2 Q(x)\in\mathcal{H}^t$, for all $t>2/3$. $\diamond$

Remark 3.4 - It is worth noting that (38) is sharp in the sense that, in general, it cannot be replaced by

$$\nabla p=O(r^{-1-\epsilon}), \tag{54}$$

for some positive ϵ. Indeed, the pairs

$$\begin{aligned} &u_r=\frac{\gamma}{r},\quad u_\theta=\alpha\left(\frac{1}{r}-r^{\gamma+1}\right),\\ &p=-\frac{\gamma^2+\alpha^2}{2r^2}-\frac{2\alpha^2r^\gamma}{\gamma}+\frac{\alpha^2}{2\gamma+2}r^{2(\gamma+1)}, \end{aligned} \tag{55}$$

with γ and α arbitrary constants, $\gamma+1\neq 0$, define the *Hamel solutions* (1916) of the Navier–Stokes equations (see [20] p. xi). For $\gamma+1=-\epsilon/4<0$ (55) is a D–solution which does not satisfy (54).

For $\gamma<-1$ (55) gives a family of D–solutions of the Navier–Stokes problem in $\complement C_1$ with boundary datum

$$u_r=\gamma,\quad u_\theta=0. \tag{56}$$

Therefore, at least for $\gamma+1<0$, problem (2) with the condition at infinity

$$\lim_{r\to+\infty}\boldsymbol{u}(x)=\boldsymbol{0}$$

does not admit a uniqueness theorem in the class of D–solutions. Let us recall that if $\varphi\in D^{1,q}(\complement C_{R_0})$, $q\in[1,2)$, then there is a constant φ_0 such that (see [11] Lemma II.5.2)

$$\int_0^{2\pi}|\varphi(r,\theta)-\varphi_0|^q\le\frac{c(q)}{R^{2-q}}\int_{\complement C_{R_0}}|\nabla\varphi|^q.$$

Therefore (55) shows that also (37) is sharp in the sense that it cannot be replaced by $p\in D^{1,q}(\complement C_{R_0})$ for some $q<2$. Moreover, in contrast with (53)

a D–solution vanishing at infinity and with nonzero outflow cannot belong to any $D^{1,q}(\complement C_{R_0})$ for $q < 2$.

Note that

$$\gamma + 1 < 0 \ \Rightarrow\ |\gamma| = \frac{1}{2\pi}\left|\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}\right| > 1 \tag{57}$$

so that for $\Omega' = C_1$ the solution of Theorem 6 is not a Hamel solution. $\diamond$

Remark 3.5 - From Theorem 3 it follows that

$$\partial_r \int_0^{2\pi} u_r^2(r,\theta) = O(r^{-1}\log r) \tag{58}$$

and if $\boldsymbol{u}$ is bounded, then[15]

$$\partial_r \int_0^{2\pi} u_r^2(r,\theta) = o(r^{-1}). \tag{59}$$

Indeed, in the polar coordinate system (r,θ) $(2)_{1,2}$ read

$$\begin{aligned} \partial_r p + u_r\partial_r u_r + \frac{u_\theta}{r}\partial_\theta u_r - \frac{u_\theta^2}{r} &= 0\\ \frac{1}{r}\partial_\theta p + u_r\partial_r u_\theta + \frac{u_\theta}{r}\partial_\theta u_\theta + \frac{u_r u_\theta}{r} &= 0\\ \frac{u_r}{r} + \partial_r u_r + \frac{1}{r}\partial_\theta u_\theta &= 0. \end{aligned} \tag{60}$$

Integrating (60) over $\theta \in (0,2\pi)$ and taking into account $(60)_3$, we get

$$\partial_r \int_0^{2\pi} (p + u_r^2)(r,\theta) = \frac{1}{r}\int_0^{2\pi} (u_\theta^2 - u_r^2)(r,\theta) \tag{61}$$

Hence (58) follows by (31) and (38).

Multiply (60) by r and integrate over $C_R \setminus C_{R_0}$. Then, we have

$$\frac{1}{R}\int_{C_R\setminus C_{R_0}} \left(\frac{p+u_\theta^2}{r}\right) = \int_0^{2\pi} (p+u_r^2)(R,\theta) - \frac{R_0}{R}\int_0^{2\pi}(p+u_r^2)(R_0,\theta). \tag{62}$$

If $\boldsymbol{u}$ is bounded, then $\boldsymbol{u}(r,\theta)$ tends uniformly (in θ) to a constant vector as $r \to +\infty$. Then (62) implies that

$$\lim_{r\to+\infty} u_r^2(r,\theta) = \lim_{r\to+\infty} u_\theta^2(r,\theta)$$

and (59) folllows from (61), taking into account (38). $\diamond$

[15] By the example in Remark 3.4 relation (58) is sharp if $\boldsymbol{u} = o(1)$.

4 Existence theorems

We are now in a position to prove our general existence theorems of a D–solution for problem (2).

Theorem 4. *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^2$ and let*

$$\boldsymbol{a} \in L^2(\partial\Omega). \tag{63}$$

If

$$\frac{1}{2\pi}\left|\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}\right| < 1, \tag{64}$$

then system (2) has a Leray solution $(\boldsymbol{u}, p) \in D^{1,2}_{\delta}(\Omega) \times L^2_{\delta,\mathrm{loc}}(\overline{\Omega})$ such that (30) holds uniformly and

$$\boldsymbol{u} = \boldsymbol{u}_0 + o(1), \tag{65}$$

with $\boldsymbol{u}_0$ constant vector; it satisfies (38), (39) and if $\boldsymbol{a}$ and/or $\partial\Omega$ are more regular, then so does $(\boldsymbol{u}, p)$ according to the regularity results (i) – (iv) for the solutions of the Stokes problem; in particular, if $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 2)$, then $\boldsymbol{u} \xrightarrow{\mathrm{nt}} \boldsymbol{a}$. Moreover, there are positive constants ϵ and $\mu_0 < 1$ depending on Ω such that

- (j) *if $\boldsymbol{a} \in C^{0,\mu}(\partial\Omega)$, then $\boldsymbol{u} \in C^{0,\mu}_{\mathrm{loc}}(\overline{\Omega})$ for $\mu \in [0, \mu_0)$;*
- (jj) *if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q \in (\max\{4/3, 2-\epsilon\}, 2+\epsilon)$, then $(\boldsymbol{u}, p) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}) \times L^q_{\mathrm{loc}}(\overline{\Omega})$; if Ω is of class C^1 we can take $\mu_0 = 1$ and $q \in [4/3, +\infty)$;*
- (jjj) *$\boldsymbol{a} \in W^{1,2}(\partial\Omega)$, then $(\boldsymbol{u}, p) \in D^{2,2}_{\delta}(\Omega) \times D^{1,2}_{\delta}(\Omega)$.*

Proof – We look for a solution of (2) in the form $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{h}$, with $\boldsymbol{w} \in D^{1,2}_{\sigma,0}(\Omega)$ and $\boldsymbol{h}$ defined by (17). As is well–known [4], [33], [34], under assumption (64) the system

$$\begin{aligned}
\Delta \boldsymbol{w} - (\boldsymbol{h} + \boldsymbol{w})\cdot\nabla(\boldsymbol{h} + \boldsymbol{w}) + \Delta\boldsymbol{\zeta} &= \nabla Q \qquad \text{in } \Omega_k,\\
\operatorname{div} \boldsymbol{w} &= 0 \qquad \text{in } \Omega_k,\\
\boldsymbol{w} &= \boldsymbol{0} \qquad \text{on } \partial\Omega_k
\end{aligned} \tag{66}$$

(for all $k > k_0 > R_0$) has a solution $\boldsymbol{w}_k \in W^{1,2}_{\sigma,0}(\Omega_k)$ we extend to all $\mathbb{R}^2$ by setting $\boldsymbol{w}_k = \boldsymbol{0}$ in $\complement\Omega$. Of course, $\boldsymbol{w}_k$ satisfies the equation

$$\int_{\Omega} \nabla \boldsymbol{w}_k \cdot \nabla\boldsymbol{\varphi} = \int_{\Omega} (\boldsymbol{h} + \boldsymbol{w}_k)\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{h} + \boldsymbol{w}_k) - \int_{\Omega} \nabla\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\varphi}, \tag{67}$$

for all $\boldsymbol{\varphi} \in W^{1,2}_{\sigma,0}(\Omega_k)$.

Let us show that if (64) holds, then there is a positive number c_0 independent of k such that

$$\int_\Omega |\nabla \boldsymbol{w}_k|^2 \le c_0. \tag{68}$$

To prove (68) we use a well–known reasoning of J. Leray (see also [4] and [11] section VIII.7). If (68) is not true, then we can find a sequence of solutions $\{\boldsymbol{w}'_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} J_k^2 = \lim_{k\to+\infty} \int_\Omega |\nabla \boldsymbol{w}'_k|^2 = +\infty.$$

In virtue of (67) the field

$$\boldsymbol{w}_k = \frac{\boldsymbol{w}'_k}{J_k}$$

satisfies

$$\begin{aligned} \frac{1}{J_k}\int_\Omega \nabla\boldsymbol{\varphi}\cdot\nabla\boldsymbol{w}_k &= \int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{w}_k + \frac{1}{J_k}\int_\Omega \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{w}_k \\ &\quad + \frac{1}{J_k}\int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} + \frac{1}{J_k^2}\int_\Omega (\boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} - \nabla\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\varphi}), \end{aligned} \tag{69}$$

for all $\boldsymbol{\varphi} \in W^{1,2}_{0,\sigma}(\Omega_k)$. Since $\|\nabla\boldsymbol{w}_k\|_{L^2(\Omega)} = 1$, by the compactness theorem of F. Rellich from $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ we can extract a subsequence, we denote by the same symbol, which converges strongly in $L^q_{\rm loc}(\overline{\Omega})$, for all $q \in (1,+\infty)$, and weakly in $D^{1,2}_0(\Omega)$ to a field $\boldsymbol{w} \in D^{1,2}_{\sigma,0}(\Omega)$, with $\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le 1$. Letting $k \to +\infty$ in (69), we see that the field $\boldsymbol{w}$ is a weak solution of the Euler equations

$$\begin{aligned} \boldsymbol{w}\cdot\nabla\boldsymbol{w} + \nabla Q &= 0 \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{w} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{w} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{70}$$

for some pressure field $Q \in W^{1,q}_{\rm loc}(\overline{\Omega})$, $q \in [1,2)$, constant on $\partial\Omega$ [19]. Now, choosing $\boldsymbol{\varphi} = \boldsymbol{w}'_k$ in (69) we get

$$\begin{aligned} 1 &= \int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{w}_k\cdot\boldsymbol{\sigma} + \int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{w}_k\cdot(\boldsymbol{h}-\boldsymbol{\sigma}) \\ &\quad + \frac{1}{J_k}\int_\Omega (\boldsymbol{h}\cdot\nabla\boldsymbol{w}_k\cdot\boldsymbol{h} - \nabla\boldsymbol{w}_k\cdot\nabla\boldsymbol{\zeta}). \end{aligned} \tag{71}$$

By (22)

$$\left|\int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{w}_k\cdot\boldsymbol{\sigma}\right| \le \frac{1}{2\pi}\left|\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}\right|\int_\Omega|\nabla\boldsymbol{w}_k|^2 \le \frac{1}{2\pi}\left|\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}\right|.$$

Therefore (71) yields

$$\begin{aligned} 1-\frac{1}{2\pi}\left|\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}\right| &\le \int_\Omega \boldsymbol{w}_k\cdot\nabla\boldsymbol{w}_k\cdot(\boldsymbol{h}-\boldsymbol{\sigma}) \\ &\quad+\frac{1}{J_k}\int_\Omega(\boldsymbol{h}\cdot\nabla\boldsymbol{w}_k\cdot\boldsymbol{h}-\nabla\boldsymbol{w}_k\cdot\nabla\boldsymbol{\zeta}). \end{aligned} \tag{72}$$

Hence, taking into account that by (16)

$$\begin{gathered} \left|\int_\Omega \boldsymbol{h}\cdot\nabla\boldsymbol{w}_k\cdot\boldsymbol{h}\right| \le \left\{\int_\Omega|\boldsymbol{h}|^4\int_\Omega|\nabla\boldsymbol{w}_k|^2\right\}^{1/2} \le c, \\ \left|\int_\Omega\nabla\boldsymbol{w}_k\cdot\nabla\boldsymbol{\zeta}\right| = \left|\int_{T_{\bar R}}\nabla\boldsymbol{w}_k\cdot\nabla\boldsymbol{\zeta}\right| \le \left\{\int_\Omega|\nabla\boldsymbol{w}_k|^2\int_{T_{\bar R}}|\nabla\boldsymbol{\zeta}|^2\right\}^{1/2} \le c, \end{gathered}$$

and letting $k\to+\infty$ in (72), it follows

$$1-\frac{1}{2\pi}\left|\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}\right| \le \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{\zeta}. \tag{73}$$

Taking into account that Q is constant on $\partial\Omega$ (say Q_0) and $\boldsymbol{\zeta}$ is divergence free in $\mathbb{R}^2$ we have

$$\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{\zeta} = -\int_\Omega\boldsymbol{\zeta}\cdot\nabla Q = -Q_0\int_{\partial\Omega}\boldsymbol{\zeta}\cdot\boldsymbol{n} = 0. \tag{74}$$

Since, under assumption (64), (73) and (74) are incompatible, we conclude that (68) is true. Therefore, by the compactness theorem of F. Rellich from $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ we can extract a subsequence which converges strongly in $L^q_{\rm loc}(\overline{\Omega})$ and weakly in $D^{1,2}(\Omega)$ to a field $\boldsymbol{w}\in D^{2,1}_{\sigma,0}(\Omega)$ that a well–known argument shows to be a solution of equations (66) (see, *e.g.*, [38] Ch. 5).

(65) is proved in [12], [16], while (30), (38), (39) are consequence of the fact that $\boldsymbol{u}$ is a D–solution. As far as the boundary datum is concerned, let us note that $\boldsymbol{u}=\boldsymbol{h}+\boldsymbol{w}$ attains $\boldsymbol{a}$ in the following sense

$$\boldsymbol{h}\xrightarrow{\rm nt}\boldsymbol{a},\quad {\rm tr}_{|\partial\Omega}\boldsymbol{w}=\boldsymbol{0},$$

where $\mathrm{tr}_{|\partial\Omega}$ stands for the trace operator in the Sobolev space $D_0^{1,2}(\Omega)$. If $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 2)$ then $\boldsymbol{h} \in L^{2q}(\Omega)$ so that by well–known estimates about solution of the Stokes problem $\boldsymbol{w} \in W^{1,s}_{\mathrm{loc}}(\overline{\Omega})$, for some $s > 2$. Hence by Sobolev's lemma it follows that $\boldsymbol{w}$ is continuous in $\overline{\Omega}$ and $\boldsymbol{u} \xrightarrow{\mathrm{nt}} \boldsymbol{a}$. Of course, if $\boldsymbol{a} \in C(\partial\Omega)$, then $\boldsymbol{u} \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Moreover, (j)–(jjj) are consequence of (i)–(iv), □

It is not difficult to see that the above argument can be repeated for boundary data $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$, $q \geq 4$, provided we make use of the divergence free extension of $\boldsymbol{a}$ defined in Lemma 2 and assume that[16]

$$|\langle \boldsymbol{a}, \boldsymbol{n} \rangle| < 2\pi. \tag{75}$$

Indeed, the following theorem holds.

Theorem 5. *Let Ω be an exterior domain of $\mathbb{R}^2$ of class $C^{1,1}$. If $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$, $q \geq 4$, satisfies (75), then (2) has a D–solution*

$$\boldsymbol{u} \in L^q_{\mathrm{loc}}(\overline{\Omega}) \cap L^\infty(\complement C_{R_0}).$$

Moreover,

- *if $\boldsymbol{a} \in C^{1,\mu}(\partial\Omega)$, $\mu \in (0,1)$, then*
$$(\boldsymbol{u}, p) \in C^{1,\mu}_{\mathrm{loc}}(\overline{\Omega}) \times C^{0,\mu}_{\mathrm{loc}}(\overline{\Omega}),$$
- *if Ω is of class C^k $(k \geq 2)$ and $\boldsymbol{a} \in W^{k-1/q,q}(\partial\Omega)$, then*
$$(\boldsymbol{u}, p) \in W^{k,q}_{\mathrm{loc}}(\overline{\Omega}) \times W^{k-1,q}_{\mathrm{loc}}(\overline{\Omega}).$$

Let Ω be polar symmetric, *i.e*,

$$(x_1, x_2) \in \Omega \Rightarrow (-x_1, -x_2) \in \Omega.$$

If $\boldsymbol{a}$ is polar symmetric, *i.e*,

$$\boldsymbol{a}(\zeta) = -\boldsymbol{a}(-\zeta), \tag{76}$$

for all $\zeta \in \partial\Omega$, then the field $\boldsymbol{h}$ can be constructed polar symmetric and we can find a polar symmetric solution of (66). As a consequence, the solution $(\boldsymbol{u}, p)$ in Theorem 4 satisfies the symmetry properties

$$\begin{aligned} \boldsymbol{u}(x) &= -\boldsymbol{u}(-x), \\ p(x) &= p(-x), \end{aligned} \tag{77}$$

[16] By $\langle \boldsymbol{a}, \boldsymbol{n} \rangle$ we mean the value of the functional $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$ at $\boldsymbol{n}$.

for all $x \in \Omega$. Since by (77)$_1$

$$\int_0^{2\pi} \boldsymbol{u}(R,\theta) = \boldsymbol{0},$$

for all $R > R_0$, by Poincaré's inequality we get

$$\int_{T_R} |\boldsymbol{u}|^2 \le cR^2 \int_{T_R} |\nabla \boldsymbol{u}|^2, \tag{78}$$

with c independent of R. Therefore, by the trace theorem and (78)

$$\int_0^{2\pi} |\boldsymbol{u}|^2(R,\theta) \le c\left\{\frac{1}{R^2}\int_{T_R} |\boldsymbol{u}|^2 + \int_{T_R} |\nabla \boldsymbol{u}|^2\right\} \le c\int_{\complement C_R} |\nabla \boldsymbol{u}|^2, \tag{79}$$

with c independent of R. Hence it follows

$$\lim_{R\to+\infty} \int_0^{2\pi} |\boldsymbol{u}|^2(R,\theta) = 0. \tag{80}$$

By virtue of the results of [12], [17], (80) is sufficient to conclude that

$$\lim_{r\to+\infty} \boldsymbol{u}(r,\theta) = \boldsymbol{0}, \tag{81}$$

uniformly in θ. Therefore we can state

Theorem 6. *Let Ω be a polar symmetric exterior Lipschitz domain of $\mathbb{R}^2$. If $\boldsymbol{a} \in L^2(\partial\Omega)$ is polar symmetric and satisfies (64), then (2), (81) has a Leray solution which satisfies (ȷ)–(ȷȷȷ) and (30), (65), (38), (39). If Ω is of class $C^{1,1}$, then we can assume $\boldsymbol{a} \in W^{-1/4,4}(\partial\Omega)$.*

It is evident that (79) holds for every polar symmetric D–solution. Hence it follows

Theorem 7. *A polar symmetric D–solution tends to zero at infinity.*

The Hamel solutions (55) are polar symmetric and for $\gamma < -1$ have finite Dirichlet integrals. Since we can choose γ close to -1 as we want, we see that Theorem 7 is sharp in the sense that (at least for $\gamma < -1$) a polar symmetric solution cannot tend to zero at infinity as $r^{-\epsilon}$ for some positive ϵ. Note that by virtue of (57) these considerations do not apply to the D–solution of Theorem 6.

Remark 4.6 - Existence of a solution of (2) with less regular boundary data (say in $L^q(\partial\Omega)$ and $W^{-1/q,q}(\partial\Omega)$) have been studied by several authors for

bounded and regular domains with connected boundaries (see [1] [14], [15], [34] and the references therein). As far as Lipschitz domains are concerned, to the best of our knowledge problem (2) (with $L^q(\partial\Omega)$ data) has been considered only for bounded domains in [31], [33], [34] under a restriction on the flux, in [7] for small data and in [32] for domains symmetric with respect to the x_1 axis, a_1 pair function of x_2 and a_2 odd function of x_2. In [30] the classical Finn–Smith theorem [9] has been proved for Lipschitz domains and boundary data in $L^\infty(\partial\Omega)$. $\diamond$

As we said in the introduction, there is another technique, based on a Galerkin's type scheme and due to H. Fujita [10] to prove existence of a D–solution of (2), we shall call *Fujita solution*. It reduces the problem to find the uniform estimate

$$\int_\Omega |\nabla \boldsymbol{w}|^2 \le c,$$

for every solution $\boldsymbol{w} \in D^{1,2}_\sigma(\Omega)$ with compact support in Ω of the system[17]

$$\begin{aligned} \Delta \boldsymbol{w} - (\boldsymbol{h}+\boldsymbol{w})\cdot\nabla(\boldsymbol{h}+\boldsymbol{w}) + \Delta\boldsymbol{\zeta} &= \nabla Q \quad && \text{in } \Omega,\\ \operatorname{div} \boldsymbol{w} &= 0 && \text{in } \Omega,\\ \boldsymbol{w} &= \boldsymbol{0} && \text{on } \partial\Omega, \end{aligned} \tag{82}$$

where now $\boldsymbol{h}$ has the form (17) with $\boldsymbol{\zeta} = \nabla^\perp(g_{\delta_0}\eta)$, $\boldsymbol{v} = \nabla^\perp \eta$ and g_{δ_0} Leray–Hopf cut–off function of the regularized distance $\varrho(x)$ equal to 1 for $\varrho(x) \le c_1\delta_0$ and vanishing for $\varrho(x) \ge c_2\delta_0$. A straightforward calculation yields the relation

$$\int_\Omega |\nabla \boldsymbol{w}|^2 \le \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{h} + c(\delta_0)\left\{\int_{\partial\Omega}|\boldsymbol{a}|^2 + \left[\int_{\partial\Omega}|\boldsymbol{a}|^2\right]^2\right\}$$

By a classical procedure we have (see, *e.g.*, [11], [40])

$$\left|\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{\zeta}\right| \le \alpha(\delta_0)\int_\Omega |\nabla\boldsymbol{w}|^2,$$

with

$$\lim_{\delta_0\to 0}\alpha(\delta_0) = 0.$$

Moreover, by (22)

$$\left|\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\frac{\boldsymbol{e}_r}{r}\right| \le \int_\Omega |\nabla\boldsymbol{w}|^2.$$

[17] Clear expositions of this approach can be find in [11], [20], [40].

so that if (64) holds, then there is a constant c independent of $\boldsymbol{w}$ such that

$$\int_{\Omega} |\nabla \boldsymbol{w}|^2 \leq c \left\{ \int_{\partial\Omega} |\boldsymbol{a}|^2 + \left[\int_{\partial\Omega} |\boldsymbol{a}|^2 \right]^2 \right\}. \tag{83}$$

Therefore, taking also into account (15), we have

Theorem 8. *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^2$. If $\boldsymbol{a}$ satisfies (63) and (64) then (2) has a Fujita solution $(\boldsymbol{u}, p)$ such that*

$$\int_{\Omega} \delta |\nabla \boldsymbol{u}|^2 \leq c \left\{ \int_{\partial\Omega} |\boldsymbol{a}|^2 + \left[\int_{\partial\Omega} |\boldsymbol{a}|^2 \right]^2 \right\}. \tag{84}$$

Theorem 9. *Let Ω be an exterior domain of $\mathbb{R}^2$ of class $C^{1,1}$. If $\boldsymbol{a} \in W^{-1/4,4}(\partial\Omega)$, satisfies (75), then (2) has a D–solution $\boldsymbol{u} \in L^4_{\rm loc}(\overline{\Omega})$.*

It is quite evident that the Fujita solutions enjoys all the regularity properties as those of the Leray solution. The only substantial difference is that the latter is always bounded while by Theorem 2 we know that the former is bounded for zero outflow.

Remark 4.7 - It is not difficult to see that Theorems 2 – 9 can be stated for the system

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega \end{aligned} \tag{85}$$

in the more general exterior domain[18]

$$\Omega = \mathbb{R}^2 \setminus \overline{\Omega'}, \quad \Omega' = \bigcup_{i=1}^{m} \Omega_i, \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing, \ i \neq j, \tag{86}$$

with $\partial\Omega_i$ Lipschitz and connected, provided

$$\boldsymbol{f} \in \mathcal{H}^1(\Omega) \tag{87}$$

[18] Lemma 1 continues to hold for the domain (86); in such a case

$$\boldsymbol{\sigma} = -\frac{1}{2\pi} \sum_{i=1}^{m} \frac{(x - x_i)}{|x - x_i|^2} \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n},$$

where x_i is a fixed point of Ω_i.

vanishes outside a bounded set[19], is polar symmetric in Theorems 6, 7 and

$$\frac{1}{2\pi}\sum_{i=1}^{m}\left|\int_{\partial\Omega_i}\boldsymbol{a}\cdot\boldsymbol{n}\right|<1$$

($\sum_{i=1}^{m}|\langle\boldsymbol{a},\boldsymbol{n}\rangle|<2\pi$ for Ω of class $C^{1,1}$ and $\boldsymbol{a}\in W^{-1/4,4}(\partial\Omega)$). Under assumption (87) $(\boldsymbol{u},p)$ satisfies (85) almost everywhere in Ω and $\boldsymbol{u}$ is continuous in Ω [3]. Moreover, if $\operatorname{div}\boldsymbol{f}\in\mathcal{H}^1(\Omega)$, then p is continuous in Ω. Moreover, (84) becomes

$$\|\boldsymbol{u}\|_{D^{1,2}_{\delta}(\Omega)}\le c\left\{\|\boldsymbol{a}\|_{L^2(\partial\Omega)}+\|\boldsymbol{f}\|_{\mathcal{H}^1}+\left[\|\boldsymbol{a}\|_{L^2(\partial\Omega)}+\|\boldsymbol{f}\|_{\mathcal{H}^1}\right]^2\right\}. \qquad\diamond$$

5 A uniqueness theorem

Uniqueness of a D–solution converging to a nonzero vector at infinity[20] is a complicated question and only in few cases we know as to determine small uniqueness classes (see [9] and [11] Ch. X). We aim at observing now as uniqueness could be linked with the boundary data at least in particular situations: *the potential flows*.

Let us consider the harmonic simple layer potential with density ψ[21]

$$v(x)=\frac{1}{2\pi}\int_{\partial\Omega}\psi(\zeta)\log|x-\zeta|\mathrm{d}s_{\zeta} \tag{88}$$

and the Navier–Stokes problem

$$\begin{aligned}\Delta\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\boldsymbol{u}&=\nabla p\quad&&\text{in }\Omega,\\ \operatorname{div}\boldsymbol{u}&=0&&\text{in }\Omega,\\ \boldsymbol{u}&=\boldsymbol{a}&&\text{on }\partial\Omega,\\ \boldsymbol{u}&=\boldsymbol{e}_1+o(1),\end{aligned} \tag{89}$$

with the boundary datum

$$\boldsymbol{a}(\xi)=\nabla v(\xi)+\boldsymbol{e}_1. \tag{90}$$

[19] This is not necessary for the existence of a D–solution to (85). It is worthy to note that for the validity of Theorem 2 it is sufficient that (34) holds for a circumference surrounding $\overline{\Omega'}$.

[20] Recall that uniqueness does not hold when the D–solution is zero at infinity at least for large Reynolds numbers (see Remark 3.4).

[21] It can be proved that every harmonic function u in a Lipschitz exterior domain Ω of $\mathbb{R}^2$ such that $u=o(r)$ and $\mathrm{tr}_{|\partial\Omega}u\in W^{1,2}(\partial\Omega)$, is expressed by (88) for some $\psi\in L^2(\partial\Omega)$.

Note that by

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = \int_{\partial\Omega} \partial_n v = \int_{\partial\Omega} \psi.$$

The pair

$$(\nabla v + \boldsymbol{e}_1, -\tfrac{1}{2}|\nabla v|^2 - \partial_1 v) \tag{91}$$

is a D–solution to (89), (90). By what we said above, we could have other *a priori different* D–solutions of (89)–(90), as the Finn–Smith, Galdi, Leray and Fujita solutions. Let shows that if

$$\int_{\partial\Omega} |\psi| < 2\pi, \tag{92}$$

then all these solutions coincide. Indeed, the following theorem holds.

Theorem 10. *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^2$. If $\psi \in L^2(\partial\Omega)$ satisfies (92), then (91) is unique in the class of all D–solutions.*

Proof – Let $(\boldsymbol{u}+\boldsymbol{w}, p+Q)$ be another D–solution to (2), (90). Then $(\boldsymbol{w}, Q)$ satisfies the equation

$$\begin{aligned} \Delta\boldsymbol{w} - (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{u} &= \nabla Q \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{w} &= 0 \qquad \text{in } \Omega,\\ \boldsymbol{w} &= \boldsymbol{0} \qquad \text{on } \partial\Omega,\\ \boldsymbol{w} &= o(1). \end{aligned} \tag{93}$$

Let $g(r)$ be a regular function, equal to 1 in C_R, vanishing outside C_{2R} and such that $|\nabla g| \le cR^{-1}$. Then by a standard computation we get

$$\begin{aligned} \int_\Omega g|\nabla\boldsymbol{w}|^2 = &\int_{T_R} \left[\tfrac{1}{2}|\boldsymbol{w}|^2(\boldsymbol{u}+\boldsymbol{w}) + (\boldsymbol{u}\cdot\boldsymbol{w}+Q)\boldsymbol{w}\right]\cdot\nabla g\\ &+ \int_\Omega g\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}. \end{aligned} \tag{94}$$

By Hölder's inequality and (53)

$$\left|\int_\Omega |\boldsymbol{w}|^2(\boldsymbol{u}-\boldsymbol{e}_1)\cdot\nabla g\right| \le \frac{c}{R}\left\{\int_{T_R}|\boldsymbol{u}-\boldsymbol{e}_1|^4\right\}^{1/4}\left\{\int_{T_R}|\boldsymbol{w}|^8\right\}^{1/4}\left\{\int_{T_R}\right\}^{1/2} = o(1),$$

$$\left|\int_\Omega |\boldsymbol{w}|^2\partial_1 g\right| \le \frac{c}{R}\left\{\int_{T_R}|\boldsymbol{w}|^4\right\}^{1/2}\left\{\int_{T_R}\right\}^{1/2} = o(1),$$

$$\left|\int_\Omega |\boldsymbol{w}|^2\boldsymbol{w}\cdot\nabla g\right| \le \frac{c}{R}\left\{\int_{T_R}|\boldsymbol{w}|^6\right\}^{1/3}\left\{\int_{T_R}\right\}^{1/2} = o(1),$$

$$\left|\int_\Omega Q\boldsymbol{w}\cdot\nabla g\right| \le \frac{c}{R}\left\{\int_{T_R}|\boldsymbol{w}|^4\int_{T_R}|Q|^4\right\}^{1/4}\left\{\int_{T_R}\right\}^{1/2} = o(1).$$

Likewise,

$$\left|\int_\Omega (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{w}\cdot\nabla g\right| = o(1).$$

Moreover,

$$\left|\int_\Omega g\boldsymbol{w}\cdot\nabla\boldsymbol{w}_1\right| = \left|\int_{T_R} w_1\boldsymbol{w}\cdot\nabla g\right| \le \frac{c}{R}\left\{\int_{T_R}|\boldsymbol{w}|^4\right\}^{1/2}\left\{\int_{T_R}\right\}^{1/2} = o(1)$$

By (22)

$$\begin{aligned}\left|\int_\Omega \boldsymbol{w}\cdot\ \nabla\boldsymbol{w}\cdot\nabla v\right| &= \left|\int_\Omega v\nabla\boldsymbol{w}\cdot\nabla\boldsymbol{w}^{\mathrm{T}}\right| \\ &= \frac{1}{2\pi}\left|\int_{\partial\Omega}\psi(\zeta)\int_\Omega \nabla\boldsymbol{w}\cdot\nabla\boldsymbol{w}^{\mathrm{T}}\log|x-\zeta|\mathrm{d}a_x\right| \\ &= \frac{1}{2\pi}\left|\int_{\partial\Omega}\psi(\zeta)\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\frac{(x-\zeta)}{|x-\zeta|^2}\mathrm{d}a_x\right| \le \frac{\|\psi\|_{L^1(\partial\Omega)}}{2\pi}\int_\Omega|\nabla\boldsymbol{w}|^2.\end{aligned}$$

for all $\boldsymbol{w}\in D^{1,2}_{\sigma,0}(\Omega)$. Therefore, letting $R\to+\infty$ in (94), we have

$$\left(2\pi-\|\psi\|_{L^1(\partial\Omega)}\right)\int_\Omega|\nabla\boldsymbol{w}|^2 \le 0.$$

Hence uniqueness follows at once. □

Remark 5.8 - Note that if

$$v(x) = \frac{\mu\log r}{2\pi},$$

then

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = \mu$$

and (92) takes the weaker form.

$$|\mu| < 2\pi. \qquad \diamond$$

Remark 5.9 - When $\partial\Omega$ is connected and $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{u}_0 = \boldsymbol{e}_1$, a solution of the equations

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nabla p \quad &&\text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 &&\text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} &&\text{on } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) &= \boldsymbol{e}_1 \end{aligned} \tag{95}$$

represents the translational motion (with velocity $-\boldsymbol{e}_1$) of an object in a Navier–Stokes fluid assumed to be at rest at infinity. As we remarked in this paper, problem (95) is completely open. By the Leray argument we know that the sequence of solutions of the systems

$$\begin{aligned} \Delta \boldsymbol{u}_k - \boldsymbol{u}_k \cdot \nabla \boldsymbol{u}_k &= \nabla p_k \quad &&\text{in } \Omega_k, \\ \operatorname{div} \boldsymbol{u}_k &= 0 &&\text{in } \Omega_k, \\ \boldsymbol{u}_k &= \boldsymbol{0} &&\text{on } \partial\Omega_k, \\ \boldsymbol{u}_k &= \boldsymbol{e}_1 &&\text{on } \partial C_k \end{aligned} \tag{96}$$

converges to a D–solution to $(95)_{1,2,3}$ and there is a constant vector $\boldsymbol{u}_0$ such that [16]

$$\lim_{r \to +\infty} \boldsymbol{u}(r, \theta) = \boldsymbol{u}_0,$$

uniformly on θ. However, we do not know $\boldsymbol{u}_0$ so that in principle it could be zero and the Leray construction could even yield the trivial solution, as it happens for the Stokes paradox (see Section 6). C.J. Amick excluded this possibility for domains of class C^3, symmetric with respect to the x_1–axis [2] (see also [12]). This result has been recently extended to symmetric Lipschitz domains in [29].

6 Some remarks on the Stokes paradox

More in general, introducing the Reynolds number $\lambda = vl/\nu$, with v, l reference velocity and reference length, and ν kinematical viscosity of the fluid, the steady–state Navier–Stokes problem in an exterior Lipschitz domain Ω of $\mathbb{R}^2$ writes

$$\begin{aligned} \Delta \boldsymbol{u} - \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0. & \end{aligned} \tag{97}$$

Of course, for $\lambda = 0$ (97) reduces to the Stokes problem

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0. & \end{aligned} \tag{98}$$

In this section we aim at comparing the known results for systems (97), (98). It is well–known a D–solution of $(98)_{1,2,3}$ exists and converges to a constant vector, but contrary to what happens in the nonlinear case, we know that (98) has a solution if and only if $\boldsymbol{a} \in L^2(\partial\Omega)$, $\boldsymbol{f} \in \mathcal{H}^1$ and $\boldsymbol{u}_0$ satisfy the compatibility condition [13], [34]

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{T}(\boldsymbol{h}_i, p_i) \cdot \boldsymbol{n} + \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{h}_i = 0, \quad i = 1, 2, \tag{99}$$

with $(\boldsymbol{h}_i, p_i)$ solution of[22]

$$\begin{aligned} \Delta \boldsymbol{h}_i = \nabla p_i \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{h}_i = 0 \quad & \text{in } \Omega, \\ \boldsymbol{h}_i = \boldsymbol{e}_i \quad & \text{on } \partial\Omega, \\ \boldsymbol{h}_i = o(r). & \end{aligned} \tag{100}$$

For instance, if $\partial\Omega$ is an ellipse, then $[\boldsymbol{T}(\boldsymbol{h}_i, p_i) \cdot \boldsymbol{n}](\xi) = (\boldsymbol{\xi} \cdot \boldsymbol{n}(\xi))\boldsymbol{e}_i$ [23] and for $\boldsymbol{f} = \boldsymbol{0}$ (99) writes

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0)_i (\boldsymbol{\xi} \cdot \boldsymbol{n}(\xi)) = 0, \quad i = 1, 2.$$

[22] The solutions of (100) span a linear space of dimension two and every $\boldsymbol{h}$ behaves at infinity as $\log r$

In particular, if $\boldsymbol{u}$ is a D–solution of (98)$_{1,2,3}$, then for large R

$$\frac{1}{2\pi}\int_0^{2\pi} u_i(R,\theta)\mathrm{d}\theta$$

is constant and gives the vector to which $\boldsymbol{u}$ tends at infinity (Picone's mean theorem at infinity).

Since

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{h}_i, p_i)\cdot\boldsymbol{n} \neq \boldsymbol{0},$$

from (99) it follows that if $\boldsymbol{a} = \boldsymbol{f} = \boldsymbol{0}$, then the only solution to (98) is the trivial one so that necessarily $\boldsymbol{u}_0$ must be zero (Stokes' paradox). The results of R. Finn & D.R. Smith [9] (see also [11], [30]) and C.J. Amick [2] (see Remark 5.9) allow us to state

Theorem 11. *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^2$. If λ is sufficiently small or Ω is symmetric with respect to an axis, then the Stokes paradox holds if and only if $\lambda = 0$.*

Of course, a D–solution of (97) must satisfy (99) whenever the integrals

$$\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{h}$$

make sense. In particular, taking into account that if Ω is polar symmetric, then $\boldsymbol{h}(x) = \boldsymbol{h}(-x)$ for all $x \in \Omega$, we see that the solution of Theorem 6 satisfy (99).

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